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bridged the gap between analytic geometry and function-theory, by way of the fourth dimension. There are many other ways of graphing imaginaries, no doubt; but the above is to be preferred, because it is a logical and systematic scheme resting at both ends on the approved processes.

AN INTRINSIC EQUATION SOLUTION OF A PROBLEM OF EULER.

By PAUL R. RIDER, Washington University.

In his *Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes*, Lausanne (1744), p. 64, Euler proposed and solved the following problem: Given two points P_0 and P_1, and directed lines P_0Q and QP_1 through them, to determine an arc tangent to these two lines at P_0 and P_1, which with its evolute and its normals at P_0 and P_1 will enclose the minimum area. This area is of course expressed by the definite integral $\int_{s_0}^{s_1} \frac{1}{2}\rho ds$, in which ρ is the radius of curvature. The ordinary method of solution is to express the integral in the form

$$\int_{x_0}^{x_1} \frac{(1+y'^2)^2}{2y''}\,dx, \tag{1}$$

and to find, by methods of the calculus of variations, the form of the curves that will give it a minimum value. (It is to be observed that y'', the second derivative of y with respect to x, occurs in the integrand. The problem is of particular interest on this account.) It is necessary to make some substitutions and transformations that are not altogether simple, before it is evident that the minimizing curves are cycloids.[1]

I wish to give here a shorter solution by obtaining the intrinsic equation of the extremals (*i. e.*, minimizing curves). The intrinsic equation of a curve is defined as the relation between s, the length of arc measured from a fixed point on it, and ϕ, the angle of deviation of the tangent at any point from the tangent at the fixed point taken as origin. This is in the sense of Whewell.[2]

If the area integral (1) is written in the form

$$\int_{s_0}^{s_1} \frac{ds}{2\kappa}. \tag{2}$$

where κ is the curvature, we have a special case of an integral studied by Radon,[3] namely,

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[1] See, for example, Euler, *loc. cit.*, or the solution outlined by Bolza, *Vorlesungen über Variationsrechnung*, p. 152.

[2] See *Cambridge Philosophical Transactions*, Vols. 8 and 9; or Williamson, *Differential Calculus*, 9th edition, revised (1899), p. 304.

[3] Radon, "Über das Minimum des Integrals $\int_{s_0}^{s_1} F(x, y, \theta, \kappa)ds$," *Sitzungsberichte der kaiserlichen Akademie der Wissenschaften mathematisch-naturwissenschaftliche Klasse*, Wien, Vol. 119 (1910), pp. 1257–1326.

$$\int_{s_0}^{s_1} F(x, y, \theta, \kappa)ds, \tag{3}$$

where θ is the angle that the tangent line at any point of a curve makes with the x-axis, and κ is the curvature of the curve at the given point. He developed the theory of the calculus of variations for this type of integral, and discussed in detail the case in which F is a function of κ alone. He even mentioned the Euler problem, but did not give a new solution of it, although as will be seen, the intrinsic equation of the extremals can readily and neatly be derived from his results.

For the integral (3) the so-called Euler equations, or differential equations that must be satisfied by minimizing curves, were discovered by Radon to be

$$\begin{aligned} &F_x - \frac{d}{ds}\left[\cos\theta(F - \kappa F_\kappa) - \sin\theta\left(F_\theta - \frac{dF_\kappa}{ds}\right)\right] = 0, \\ &F_y - \frac{d}{ds}\left[\sin\theta(F - \kappa F_\kappa) + \cos\theta\left(F_\theta - \frac{dF_\kappa}{ds}\right)\right] = 0, \end{aligned} \tag{4}$$

subscripts denoting partial differentiation. For the integral (2),

$$F = \frac{1}{2\kappa}, \qquad F_x = F_y = F_\theta = 0, \qquad F_\kappa = -\frac{1}{2\kappa^2},$$

and equations (4) reduce to

$$-\frac{d}{ds}\left[\cos\theta\left(\frac{1}{2\kappa} - \kappa\left\{-\frac{1}{2\kappa^2}\right\}\right) - \sin\theta\left(-\frac{d}{ds}\left\{-\frac{1}{2\kappa^2}\right\}\right)\right] = 0,$$

$$-\frac{d}{ds}\left[\sin\theta\left(\frac{1}{2\kappa} - \kappa\left\{-\frac{1}{2\kappa^2}\right\}\right) + \cos\theta\left(-\frac{d}{ds}\left\{-\frac{1}{2\kappa^2}\right\}\right)\right] = 0.$$

Integration gives

$$\cos\theta\cdot\frac{1}{\kappa} - \sin\theta\cdot\frac{d}{ds}\frac{1}{2\kappa^2} = \alpha,$$

$$\sin\theta\cdot\frac{1}{\kappa} + \cos\theta\cdot\frac{d}{ds}\frac{1}{2\kappa^2} = \beta.$$

Multiplying the first of these equations by $\cos\theta$, the second by $\sin\theta$, and adding, we obtain

$$\frac{1}{\kappa} = \alpha\cos\theta + \beta\sin\theta,$$

or, since $\kappa = \dfrac{d\theta}{ds}$,

$$ds = (\alpha\cos\theta + \beta\sin\theta)d\theta,$$

from which

$$s - s_0 = \alpha\sin\theta - \beta\cos\theta.$$

Let $\alpha = a \cos \theta_0$, $\beta = a \sin \theta_0$. Then

$$s - s_0 = a \sin (\theta - \theta_0).$$

But this is the intrinsic equation of the cycloid.[1]

BOOK REVIEWS.

SEND ALL COMMUNICATIONS TO W. H. BUSSEY, University of Minnesota.

An Elementary Course in Synthetic Projective Geometry. By DERRICK NORMAN LEHMER. Ginn and Company, Boston, 1917. xiii + 123 pages.

Lehmer's *Projective Geometry* "is intended to give in as simple a way as possible the essentials of synthetic projective geometry." The author has done this in the 74 pages of Chapters I—VI, VIII by the methods of pure geometry without the use of anharmonic ratios, circles or any other metric notions. It is his idea that "a purely projective notion ought not to be based on metrical foundations." "The course is not intended to furnish an illustration of how a subject may be developed from the smallest possible number of fundamental assumptions. The author is aware of the importance of work of this sort, but he does not believe it is possible at the present time to write a book along such lines which shall be of much use for elementary students." For the purpose of this course the student should have a thorough knowledge of high-school plane geometry and enough solid geometry to understand the proof of Desargues's theorem about perspective triangles: "*If two triangles ABC and $A'B'C'$ are so situated that the lines AA', BB', CC' all meet in a point, then the pairs of sides AB and $A'B'$, BC and $B'C'$, CA and $C'A'$ all meet on a straight line, and conversely.*"

The first chapter of the book is devoted to the important notion of one-to-one correspondence. In it is explained the need of the fiction about points and lines at infinity. On page 7, after some remarks on one-to-one correspondences being continuous, the author says: "In the case of point-rows this continuity is subject to exception in the neighborhood of the point 'at infinity.'" The reviewer doubts that the student will understand what the "neighborhood of the point at infinity" is.

The theorem of Desargues is very easily proved when the two triangles are not in one plane. When they are in one plane the usual method of proof consists in showing that a third triangle $A''B''C''$ can be found which is perspective to both ABC and $A'B'C'$ from two different centers of projection, and in twice making use of the theorem for two perspective triangles not in one plane. The only difficulty which the student is likely to have with this proof is in the making and the visualizing of the diagram. Lehmer proves the theorem for two triangles not in one plane in the usual way, and draws an illustrative figure;

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[1] See Williamson, *loc. cit.*, p. 338.